A moment approach to non-Gaussian colored noise

Hideo Hasegawa [1]

Department of Physics, Tokyo Gakugei University
Koganei, Tokyo 184-8501, Japan

(February 6, 2008)

Abstract

The Langevin system subjected to non-Gaussian noise has been discussed, by using the second-order moment approach with two kinds of models for generating the noise. We have derived the effective differential equation (DE) for a variable x, from which the stationary probability distribution $P(x)$ has been calculated with the use of the Fokker-Planck equation. The result of $P(x)$ calculated by the moment method is compared to several expressions obtained by different methods such as the universal colored noise approximation (UCNA) [Jung and Hänggi, Phys. Rev. A **35**, 4464 (1987)] and the functional-integral method. It has been shown that our $P(x)$ is in good agreement with that of direct simulations (DSs). We have also discussed dynamical properties of the model with an external input, solving DEs in the moment method.

PACS No. 05.40.Ca, 05.10.Gg, 05.10.-a

Phone: +81-42-329-7482

Fax: +81-42-329-7491

E-mail: hideohasegawa@goo.jp

[1] E-mail address: hideohasegawa@goo.jp

1 INTRODUCTION

Interesting, unconventional phenomena such as the stochastic resonance (SR) and the noise-induced phase transition are created by noise. Theoretical studies on noise in non-linear dynamical systems have usually adopted Gaussian white (or colored) noise. In recent years, there is a growing interest in studying dynamical systems driven by non-Gaussian noise. This is motivated by the fact that non-Gaussian noise with random amplitudes following the power-law distribution is quite ubiquitous in natural phenomena. For example, experimental results for crayfish and rat skin offer strong indication that there could be non-Gaussian noise in these sensory systems [1][2]. A simple mechanism has been proposed to generate the non-Gaussian noise [3]. With the use of such a theoretical model, the SR induced by non-Gaussian colored noise has been investigated [4]. It has been shown that the peak in the signal-to-noise ratio (SNR) for non-Gaussian noise becomes broader than that for Gaussian noise. This result has been confirmed by an analog experiment [5].

Stochastic systems with non-Gaussian colored noise are originally expressed by the non-Markovian process. This problem is transformed into a Markovian one by extending the number of variables and equations. The relevant Fokker-Planck equation (FPE) includes the probability distribution expressed in terms of multi-variables. We may transform this FPE for multivariate probability to the effective single-variable FPE, or obtain one-variable differential equation (DE) with the use of some approximation methods like the universal colored noise approximation (UCNA) [6, 7] and the functional-integral methods [8][9]. The obtained results, however, do not agree each other, depending on the adopted approximations, as will be explained in Sec. 2.2 (Table 1). It is not easy to trace the origin of this discrepancy because of the complexity in adopted procedures. The purpose of the present paper is to discuss the non-Gaussian noise and to make a comparison among various methods, by employing the second-order moment method which is simple and transparent, and which is exact in the weak-noise limit.

The paper is organized as follows. We have applied the second-moment method to the Langevin model subjected to non-Gaussian noise which is generated by two kinds of models. In Sec. 2, non-Gaussian noise is generated by the specific function which was proposed by Borland [3] and which has been adopted in several studies [4][8][9]. In contrast, in Sec. 3, non-Gaussian noise is generated by multiplicative noise [10]-[14]. We derive the effective one-variable DE, from which the stationary distribution is calculated

with the use of the FPE. A comparison among various methods generating the non-Gaussian noise is made is Sec. 4, where contributions from higher moments than the second moment are also discussed. The final Sec. 5 is devoted to our conclusion.

2 Models A_0 and A

2.1 Moment method

We have adopted the Langevin model subjected to non-Gaussian colored noise (ϵ) and Gaussian white noise ($\psi\xi$), as given by [3]

$$\dot{x} = F(x) + \epsilon(t) + \psi\xi(t) + I(t), \tag{1}$$

$$\tau\dot{\epsilon} = K(\epsilon) + \phi\eta(t), \qquad \text{(model A}_0\text{)} \tag{2}$$

with

$$K(\epsilon) = -\frac{\epsilon}{[1 + (q-1)(\tau/\phi^2)\epsilon^2]}, \tag{3}$$

which is referred to as the model A_0. In Eqs. (1)-(3), $F(x)$ is an arbitrary function of x, $I(t)$ stands for an external input, q is a parameter expressing a departure from the Gaussian distribution which is realized for $q = 1$, τ denotes the characteristic time of colored noise, and η and ξ the zero-mean white noises with correlations: $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$, $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$ and $\langle\eta(t)\xi(t')\rangle = 0$.

First, we briefly discuss the non-Gaussian colored noise generated by Eqs. (2) and (3), which yield the stationary distribution given by [3] [15, 16]

$$p_q(\epsilon) \propto \left[1 + (q-1)\left(\frac{\tau}{\phi^2}\right)\epsilon^2\right]_+^{-\frac{1}{q-1}}, \tag{4}$$

with $[x]_+ = x$ for $x \geq 0$ and zero otherwise. For $q = 1$, Eq. (3) reduces to

$$K(\epsilon) = -\epsilon, \tag{5}$$

which leads to the Gaussian distribution given by

$$p_1(\epsilon) \propto e^{-(\tau/\phi^2)\epsilon^2}. \tag{6}$$

For $q > 1$ and $q < 1$, Eq. (4) yields long-tail and cut-off distributions, respectively. Thus Eqs. (2) and (3) generate the Gaussian and non-Gaussian noises, depending on the value

of parameter q. Expectation values of ϵ and ϵ^2 are given by

$$\langle \epsilon \rangle = 0, \tag{7}$$

$$\langle \epsilon^2 \rangle = \frac{\phi^2}{\tau(5-3q)}, \tag{8}$$

which shows that $\langle \epsilon^2 \rangle$ diverges at $q = 5/3$.

In order to make our calculation tractable, we replace the ϵ^2 term in the denominator of Eq. (3) by its expectation value: $\epsilon^2 \simeq \langle \epsilon^2 \rangle$, to get [9]

$$K(\epsilon) \simeq -\frac{\epsilon}{r_q}, \tag{9}$$

by which Eq. (2) becomes

$$\tau \dot{\epsilon} = -\left(\frac{1}{r_q}\right) \epsilon + \phi \ \eta(t), \qquad \text{(model A)} \tag{10}$$

with

$$r_q = \frac{2(2-q)}{(5-3q)}. \tag{11}$$

A model given by Eq. (1) with Eq. (10) is hereafter referred to as the model A, which is discriminated from the model A_0 given by Eqs. (1)-(3). The solid curve in Fig. 1 expresses r_q. We note that we get $r_q = 1$ for $q = 1$, and $r_q < 1$ ($r_q > 1$) for $q < 1$ ($1 < q < 5/3$). The dashed curve will be discussed in Sec. 3.1.

Now we discuss the FPE of the distribution $p(x, \epsilon, t)$ for Eqs. (1) and (10), which are regarded as the coupled Langevin model. We get

$$\begin{aligned} \frac{\partial}{\partial t} p(x,\epsilon,t) &= -\frac{\partial}{\partial x}\{[F(x)+\epsilon+I]p(x,\epsilon,t)\} + \frac{\psi^2}{2}\frac{\partial^2}{\partial x^2}p(x,\epsilon,t) \\ &+ \frac{1}{r_q \tau}\frac{\partial}{\partial \epsilon}[\epsilon p(x,\epsilon,t)] + \frac{1}{2}\left(\frac{\phi}{\tau}\right)^2 \frac{\partial}{\partial \epsilon}\left[\epsilon \frac{\partial}{\partial \epsilon}\epsilon p(x,\epsilon,t)\right]. \end{aligned} \tag{12}$$

We define means, variances and covariances by

$$\langle x^m \epsilon^n \rangle = \int dx \int d\epsilon \ x^m \epsilon^n p(x,\epsilon,t). \qquad (m, n\text{: integer}) \tag{13}$$

By using the moment method for the coupled Langevin model [13, 14], we get their equations of motion given by

$$\frac{d\langle x \rangle}{dt} = \langle F(x) + \epsilon + I \rangle, \tag{14}$$

$$\frac{d\langle\epsilon\rangle}{dt} = -\frac{1}{r_q\tau}\langle\epsilon\rangle, \tag{15}$$

$$\frac{d\langle x^2\rangle}{dt} = 2\langle x[F(x)+\epsilon+I]\rangle + \psi^2, \tag{16}$$

$$\frac{d\langle\epsilon^2\rangle}{dt} = -\frac{2}{r_q\tau}\langle\epsilon^2\rangle + \left(\frac{\phi}{\tau}\right)^2, \tag{17}$$

$$\frac{d\langle x\epsilon\rangle}{dt} = \langle\epsilon[F(x)+\epsilon+I]\rangle - \frac{1}{r_q\tau}\langle x\epsilon\rangle. \tag{18}$$

We consider means, variances and covariance defined by

$$\mu = \langle x\rangle, \tag{19}$$

$$\nu = \langle\epsilon\rangle, \tag{20}$$

$$\gamma = \langle x^2\rangle - \langle x^2\rangle^2, \tag{21}$$

$$\zeta = \langle\epsilon^2\rangle - \langle\epsilon\rangle^2, \tag{22}$$

$$\chi = \langle x\epsilon\rangle - \langle x\rangle\langle\epsilon\rangle. \tag{23}$$

When we expand Eqs. (14)-(18) as $x = \mu + \delta x$ and $\epsilon = \nu + \delta\epsilon$ around the mean values of μ and ν, and retaining up to their second order contributions such as $\langle(\delta x)^2\rangle$, equations of motion become [13, 14]

$$\frac{d\mu}{dt} = f_0 + f_2\gamma + \nu + I(t), \tag{24}$$

$$\frac{d\nu}{dt} = -\frac{\nu}{r_q\tau}, \tag{25}$$

$$\frac{d\gamma}{dt} = 2(f_1\gamma + \phi) + \psi^2, \tag{26}$$

$$\frac{d\zeta}{dt} = -\left(\frac{2}{r_q\tau}\right)\zeta + \left(\frac{\phi}{\tau}\right)^2, \tag{27}$$

$$\frac{d\chi}{dt} = \left(f_1 - \frac{1}{r_q\tau}\right)\chi + \zeta, \tag{28}$$

with

$$f_\ell = \frac{1}{\ell!}\frac{\partial^\ell F(\mu)}{\partial x^\ell}. \tag{29}$$

When we adopt the stationary values for ν, ζ and ϕ:

$$\nu \simeq \nu_s = 0, \tag{30}$$

$$\zeta \simeq \zeta_s = \frac{r_q\phi^2}{2\tau}, \tag{31}$$

$$\chi \simeq \chi_s = \frac{r_q^2\phi^2}{2(1 - r_q\tau f_1)}, \tag{32}$$

equations of motion for μ and γ become

$$\frac{d\mu}{dt} = f_0 + f_2\gamma + I(t), \tag{33}$$

$$\frac{d\gamma}{dt} = 2f_1\gamma + \frac{r_q^2\phi^2}{(1-r_q\tau f_1)} + \psi^2, \tag{34}$$

where r_q is given by Eq. (11). It is noted that the stationary value of $\zeta_s = (2-q)\phi^2/\tau(5-3q)$ in Eq. (31) is a little different from $\langle\epsilon^2\rangle = \phi^2/\tau(5-3q)$ in Eq. (8), which is due to an introduced approximation.

We may express the effective DE for x as

$$\dot{x} = F_{eff}(x) + I_{eff}(t) + \alpha_{eff}\,\eta(t) + \psi\xi(t), \tag{35}$$

with

$$F_{eff}(x) = F(x), \tag{36}$$

$$I_{eff}(t) = I(t), \tag{37}$$

$$\alpha_{eff} = \frac{\phi_q}{\sqrt{1-\tau_q f_1}}, \tag{38}$$

$$\phi_q = r_q\phi, \tag{39}$$

$$\tau_q = r_q\tau, \tag{40}$$

from which Eqs. (33) and (34) are derived [13, 14]. Equations (35) and (38) clearly express the effect of non-Gaussian colored noise. The effective magnitude of noise α_{eff} is increased with increasing q (Fig. 1). In contrast, with increasing τ, the effective α_{eff} value is decreased for $f_1 < 0$ which is usually realized.

The FPE of $P(x,t)$ for Eq. (35) is expressed by

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial\epsilon}\{(F_{eff}+I)P(x,t)\} + \frac{1}{2}\frac{\partial}{\partial x}\left[\alpha_{eff}\frac{\partial}{\partial x}\alpha_{eff}P(x,t)\right] + \frac{\psi^2}{2}\frac{\partial^2}{\partial x^2}P(x,t), \tag{41}$$

which may be applicable to α_{eff} depending on x [*i.e.* Eqs. (50) and (59)]. The stationary distribution is given by

$$\ln P(x) = 2\int dx\left(\frac{F_{eff}+I}{\alpha_{eff}^2+\psi^2}\right) - \frac{1}{2}\ln\left(\frac{\alpha_{eff}^2+\psi^2}{2}\right). \tag{42}$$

For $F(x) = -\lambda x$, we get

$$P(x) \propto \exp\left[-\left(\frac{\lambda}{\left[\phi_q^2/(1+\lambda\tau_q)+\psi^2\right]}\right)\left(x-\frac{I}{\lambda}\right)^2\right]. \tag{43}$$

For $F(x) = ax - bx^3$, we get

$$P(x) \;\propto\; \exp\left[\frac{1}{[\phi_q^2/(1-\tau_q(a-3b\mu^2))+\psi^2]}\right]\left(ax^2 - \frac{bx^4}{2} + 2Ix\right). \tag{44}$$

2.2 Comparison with other methods

We will compare the result of the moment method with those of several analytical methods: the universal colored noise approximation (UCNA) and functional-integral methods (FI-1 and FI-2).

(a) UCNA

The universal colored noise approximation (UCNA) was proposed by Jung and Hänggi [6, 7] by interpolating between the two limits of $\tau = 0$ and $\tau = \infty$ of colored noise, and it has been widely adopted for a study of effects of Gaussian and non-Gaussian colored noises. By employing the UCNA, we may derive the effective DE for the variable x. Taking the time derivative of Eq. (1) with $\psi = 0$, and using Eq. (10) for $\dot{\epsilon}$, we get

$$\ddot{x} = F'\dot{x} + \dot{\epsilon} + \dot{I}, \tag{45}$$

$$= \left(F' - \frac{1}{\tau_q}\right)\dot{x} + \left(\frac{F+I}{\tau_q}\right) + \dot{I} + \left(\frac{r_q\phi}{\tau_q}\right)\eta. \tag{46}$$

When we neglect the $\ddot{x}$ term after the UCNA, we get the effective DE for x given by

$$\dot{x} = F_{eff}(x) + I_{eff}(t) + \alpha_{eff}\,\eta(t), \tag{47}$$

with

$$F_{eff}^U(x) = \frac{F(x)}{(1-\tau_q F')}, \tag{48}$$

$$I_{eff}^U(t) = \frac{(I+\tau_q\dot{I})}{(1-\tau_q F')}, \tag{49}$$

$$\alpha_{eff}^U = \frac{r_q\phi}{(1-\tau_q F')}, \tag{50}$$

where $F = F(x)$, $F' = F'(x)$, and τ_q and r_q are given by Eqs. (40) and (11), respectively. It is noted that α_{eff}^U given by Eq. (50) generally depends on x, yielding the multiplicative noise in Eq. (47).

For $F(x) = -\lambda x$, the stationary distribution is given by

$$P^U(x) \;\propto\; \exp\left[-\frac{\lambda(1+\lambda\tau_q)}{\phi_q^2}\left(x - \frac{I_c}{\lambda}\right)^2\right], \tag{51}$$

which agrees with the result of Eq. (43) with $\psi = 0$.

For $F(x) = ax - bx^3$, we get

$$P^U(x) \propto [1 - \tau_q(a - 3bx^2)] \exp\left(\left[\frac{a(1 - a\tau_q)x^2}{\phi_q^2} - \frac{b(1 - 4a\tau_q)x^4}{2\phi_q^2} - \frac{b^2\tau_q x^6}{\phi_q^2}\right]\right) \times \exp\left(\left[\frac{2I_c(1 - a\tau_q)x}{\phi_q^2} + \frac{2I_c b\tau_q x^2}{\phi_q^2}\right]\right), \quad (52)$$

whose functional form is rather different from that given by Eq. (44).

(b) Functional-integral method (FI-1)

Wu, Luo and Zhu [9] started from the formally exact expression for $P(x,t)$ of Eqs. (1) and (10) with $I(t) = 0$ given by

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}[F(x)P(x,t)] - \frac{\partial}{\partial x}\langle\epsilon(t)\delta(x(t) - x)\rangle - \psi\frac{\partial}{\partial x}\langle\xi(t)\delta(x(t) - x)\rangle, \quad (53)$$

where $\langle\cdot\rangle$ denotes the average over the probability $P(x,t)$ to be determined. Employing the Novikov theorem [17] and the functional-integral method, they obtained the effective FPE for $P(x,t)$ which yields Eq. (47) but with

$$F_{eff}^W(x) = F(x), \quad (54)$$

$$\alpha_{eff}^W = \frac{r_q\phi}{\sqrt{1 - \tau_q F_s'}}, \quad (55)$$

where r_q and τ_q are given by Eqs. (40) and (11), respectively, $F' = dF/dx$ and F_s *et al.* denote the steady-state values at $x = x_s$.

For $F(x) = -\lambda x$, we get

$$P^W(x) \propto \exp\left[-\frac{\lambda(1 + \lambda\tau_q)}{\phi_q^2}x^2\right]. \quad (56)$$

For $F(x) = ax - bx^3$, we get

$$P^W(x) \propto \exp\left[\frac{[1 - \tau_q(a - 3b\mu^2)]}{\phi_q^2}\left(ax^2 - \frac{bx^4}{2}\right)\right]. \quad (57)$$

(c) Functional-integral method (FI-2)

By applying the alternative functional-integral method to the FPE for $p(x,\epsilon,t)$ given by Eqs. (1) and (10) with $\psi = I(t) = 0$, Fuentes, Toral and Wio [4] derived the FPE of $P(x,t)$, which leads to the effective DE given by Eq. (47), but with

$$F_{eff}^F(x) = \frac{F}{(1 - s_q\tau F')}, \quad (58)$$

$$\alpha_{eff}^F = \frac{s_q\phi}{(1 - s_q\tau F')}, \quad (59)$$

with

$$s_q = \left[1 + (q-1) \left(\frac{\tau}{2\phi^2} \right) F^2 \right]. \tag{60}$$

We note that α_{eff}^F generally depends on x, yielding the multiplicative noise in Eq. (47).

For $F(x) = -\lambda x$, we get

$$P^F(x) \propto (1 + \lambda \tau s_q) s_q^{-[2/(q-1)+1]} \exp\left[\frac{2}{\lambda \tau (q-1) s_q} \right], \tag{61}$$

with

$$s_q = \left[1 + (q-1) \left(\frac{\tau \lambda^2}{2\phi^2} \right) x^2 \right]. \tag{62}$$

For $F(x) = ax - bx^3$, it is necessary to numerically evaluate the distribution $P(x)$ with the use of Eqs. (42) and (58)-(60).

A comparison among various methods is summarized in the Table 1. We note that the result of our moment method agrees with that of FI-1, but disagrees with those of UCNA and FI-2. The result of UCNA is not identical with that of FI-2, although they are consistent each other if the identity of $s_q = r_q$ holds, which is realized for $q = 1$ with $s_1 = r_1 = 1$.

2.3 Numerical calculations

We present some numerical examples to make a comparison with direct simulation (DS), which has been performed for Eqs. (1)-(3) by the fourth-order Runge-Kutta method with a time step of 0.01 for 1000 trials. Figures 2(a)-2(f) show the stationary probability calculated by various methods for $F(x) = -\lambda x$ with changing parameters of q and τ for fixed $\phi = 0.5$ and $\psi = 0$. A comparison between Fig. 2(c) and 2(d) shows that the width of the distribution for $\tau = 1.0$ is narrower than that for $\tau = 0.5$. This is explained by the reduced effective strength of $\alpha_{eff} = \phi/(1 + \lambda\tau)$ by an increased τ. We note that for $q = 1.0$, results of all methods are in good agreement each other. Comparing Fig. 2(a) to Fig. 2(c) [and Fig. 2(b) to Fig. 2(d)], we note that the width of the distribution for $q = 0.8$ is a little narrower than that for $q = 1.0$. This is due to the fact that the r_q value is reduced to 0.82 from unity. An agreement among various methods is good for $q = 0.8$. In contrast, Figs. 2(e) and 2(f) show that for $q = 1.5$, the width of $P(x)$ becomes wider because of the increased $r_{1.5} = 2.0$. The results of the moment method, UCNA and FI-1 are in fairly good agreement. On the contrary, the distribution calculated by FI-2 is sharper than that of DS.

Figures 3(a)-3(f) show the stationary probability calculated by various methods for $F(x) = x - x^3$ with changing parameters of q and τ for fixed $\phi = 0.5$ and $\psi = 0.0$. The general trend realized in Figs. 3(a)-3(f) is the same as in Figs.2(a)-2(f). The result of FI-2 for $q = 1.5$ is not so bad compared to those of other approximation methods. However, the result of FI-2 for $q = 0.8$ and $\tau = 1.0$ is worse than other methods.

3 Model B

3.1 Moment method

In order to generate non-Gaussian noise, we may employ an alternative model (referred to as the model B) given by

$$\dot{x} = F(x) + \epsilon(t) + I(t), \tag{63}$$

$$\tau\dot{\epsilon} = -\epsilon + \epsilon\,\alpha\eta(t) + \beta\xi(t), \qquad \text{(model B)} \tag{64}$$

where $F(x)$ expresses an arbitrary function of x, $I(t)$ an external input, τ the characteristic time of colored noise, and α and β denote magnitudes of additive and multiplicative noises, respectively, given by zero-mean white noises, η and ξ, with correlations: $\langle\eta(t)\eta(t')\rangle = \langle\xi(t)\xi(t')\rangle = \delta(t-t')$ and $\langle\eta(t)\xi(t')\rangle = 0$.

The FPE for the distribution $p(\epsilon, t)$ for Eq. (64) in the Stratonovich representation is given by

$$\frac{\partial}{\partial t}p(\epsilon,t) = \frac{1}{\tau}\frac{\partial}{\partial\epsilon}[\epsilon p(\epsilon,t)] + \frac{1}{2}\left(\frac{\alpha}{\tau}\right)^2\frac{\partial}{\partial\epsilon}\left(\epsilon\frac{\partial}{\partial\epsilon}[\epsilon p(\epsilon,t)]\right) + \frac{1}{2}\left(\frac{\beta}{\tau}\right)^2\frac{\partial^2}{\partial\epsilon^2}p(\epsilon,t). \tag{65}$$

The stationary distribution of ϵ has been extensively discussed [10]-[14] in the context of the nonextensive statistics [15, 16]. It is given by [10]-[14]

$$p_q(\epsilon) \propto \left[1 + \left(\frac{\alpha^2}{\beta^2}\right)\epsilon^2\right]_+^{-(\tau/\alpha^2+1/2)}, \tag{66}$$

$$\propto \left[1 + (q-1)\left(\frac{\tau}{\kappa\beta^2}\right)\epsilon^2\right]_+^{-\frac{1}{q-1}}, \tag{67}$$

with

$$q = 1 + \left(\frac{2\alpha^2}{2\tau+\alpha^2}\right), \tag{68}$$

$$\kappa = \left(\frac{3-q}{2}\right) = \left(\frac{2\tau}{2\tau+\alpha^2}\right), \tag{69}$$

where $[x]_+ = x$ for $x \geq 0$ and zero otherwise. In the limit of $\alpha = 0.0$ ($q = 1$), the distribution given by Eq. (67) reduces to the Gaussian distribution given by

$$p(\epsilon) \;\propto\; \exp\left(-\frac{\tau}{\beta^2}\epsilon^2\right). \tag{70}$$

In the opposite limit of $\beta = 0.0$, Eq. (67) leads to the power-law distribution given by

$$p(\epsilon) \;\propto\; \epsilon^{-\delta}, \tag{71}$$

with

$$\delta \;=\; 1 + \frac{2\tau}{\alpha^2} = \frac{2}{q-1}. \tag{72}$$

The expectation values of ϵ and ϵ^2 are given by

$$\langle \epsilon \rangle \;=\; 0, \tag{73}$$

$$\langle \epsilon^2 \rangle \;=\; \frac{\kappa\beta^2}{\tau(5-3q)} = \frac{\beta^2}{2(\tau-\alpha^2)}. \tag{74}$$

The second moment is finite for $\alpha^2 < \lambda$ ($q < 5/3$). It is expected that Eq. (64) leads to the non-Gaussian colored noise with the correlation given by

$$\langle \epsilon(t)\epsilon(t') \rangle = \frac{\beta^2}{2(\tau-\alpha^2)} \exp\left[-\frac{\mid t-t' \mid}{\tau}\right]. \tag{75}$$

By applying the moment method to Eqs. (63) and (64), we may obtain the effective one-variable DE for x given by

$$\dot{x} \;=\; F_{eff} + I(t) + \beta_{eff}\,\xi(t), \tag{76}$$

with

$$F_{eff} \;=\; F(x), \tag{77}$$

$$\beta_{eff} \;=\; \frac{\beta_q}{\sqrt{1-\tau f_1}}, \tag{78}$$

$$\beta_q \;=\; \beta u_q, \tag{79}$$

$$u_q \;=\; \sqrt{\frac{1}{(1-\alpha^2/\tau)}} = \sqrt{\frac{3-q}{5-3q}}, \tag{80}$$

details of calculations being explained in the Appendix. The q dependence of u_q is plotted by the dashed curve in Fig. 1, where $u_q < 1$, $u_q = 1$ and $u_q > 1$ for $q < 1$, $q = 1$ and $1 < q < 5/3$, respectively. We note that u_q has a similar q dependence as r_q shown by the solid curve.

The FPE of $P(x,t)$ for Eq. (76) is given by

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial \epsilon}\left[(F_{eff}+I)P(x,t)\right] + \left(\frac{1}{2}\right)\frac{\partial}{\partial x}\left[\beta_{eff}\frac{\partial}{\partial x}\beta_{eff}P(x,t)\right]. \tag{81}$$

The stationary distribution is given by

$$\ln P(x) = 2\int dx \left(\frac{F_{eff}+I}{\beta_{eff}^2}\right) - \frac{1}{2}\ln\left(\frac{\beta_{eff}^2}{2}\right). \tag{82}$$

For $F(x) = -\lambda x$, we get

$$P(x) \propto \exp\left[-\left(\frac{\lambda\,(1+\lambda\tau)}{\beta_q^2}\right)\left(x-\frac{I}{\lambda}\right)^2\right]. \tag{83}$$

For $F(x) = ax - bx^3$, we get

$$P(x) \propto \exp\left[\frac{[1-\tau(a-3b\mu^2)]}{\beta_q^2}\right]\left(ax^2 - \frac{bx^4}{2} + 2Ix\right). \tag{84}$$

Equations (83) and (84) are similar to Eqs. (43) and (44) (with $\psi = 0.0$), respectively, for the model A, although u_q and τ in the former are different from r_q and τ_q in the latter.

It would be interesting to compare the result of the moment method for the model B with those of the UCNA and FI method, as we have made for the model A in Sec. 2.2. Unfortunately the UCNA method cannot be applied to the model B because Eq. (64) includes the multiplicative noise [18]. It is very difficult to apply the FI method to the model B including both additive and multiplicative noises: such calculations have not been reported as far as we are aware of. Then we will make a comparison of the result of the moment method only with that of DS in the next subsection 3.2.

3.2 Numerical calculations

We present some numerical examples to make a comparison with DS, which has been performed for Eqs. (63) and (64) by the Heun method with a time step of 0.001 for 1000 trials. Figures 4(a)-4(d) show the stationary probability $P(x)$ calculated for $F(x) = -\lambda x$ with changing parameters of α and τ for a fixed $\beta = 0.5$. In Figs. 4(a) and 4(b) for $\alpha = 0.0$ ($q = 1.0$), we observe that the width of $P(x)$ is decreased with increasing τ, as shown in Figs. 2(c) and 2(d). Figures 4(c) and 4(d) show that with increasing α to 0.5, we get wider width in $P(x)$ because we get $q = 1.40$ and 1.22 for $\tau = 0.5$ and 1.0, respectively [Eq. (68)].

Similarly, Figs. 5(a)-5(d) show $P(x)$ for $F(x) = x - x^3$. Results of the moment method are in good agreement with those of DS for $\alpha = 0.0$ ($q = 1.0$) as shown in Figs. 5(a) and

5(b). The width of $P(x)$ in Fig. 5(c) for $\alpha = 0.5$ and $\tau = 0.5$ ($q = 1.40$) is wider than that in Fig. 5(a) for $\alpha = 0.0$ and $\tau = 0.5$ ($q = 1.0$), but it is narrower than that in Fig. 5(d) for $\alpha = 0.5$ and $\tau = 1.0$ ($q = 1.22$).

4 Discussion

We will make a comparison among the various methods for generating non-Gaussian noise given by

$$\dot{x} = F(x) + \epsilon(t) + I(t), \tag{85}$$

with

$$\tau\dot{\epsilon} = K(\epsilon) + \phi\eta(t), \qquad \text{(model A}_0\text{)} \tag{86}$$

$$\tau\dot{\epsilon} = -\left(\frac{\epsilon}{r_q}\right) + \phi\eta(t), \qquad \text{(model A)} \tag{87}$$

$$\tau\dot{\epsilon} = -\epsilon + \epsilon\,\alpha\eta(t) + \beta\xi(t), \qquad \text{(model B)} \tag{88}$$

where η and ξ are white noises, r_q is given by Eq. (11), and $K(\epsilon)$ is given by Eq. (3) or

$$K(\epsilon) = -\frac{dU(\epsilon)}{d\epsilon}, \tag{89}$$

$$U(\epsilon) = \frac{\phi^2}{2\tau(q-1)} \ln\left[1 + (q-1)\left(\frac{\tau}{\phi^2}\right)\epsilon^2\right]. \tag{90}$$

Note that the model A is derived from the model A_0 with the approximation: $K(\epsilon) \simeq -\epsilon/r_q$ and $U(\epsilon) \simeq \epsilon^2/2r_q$ [Eq. (9)]. Noises in the models A_0 and A are generated by a motion under the potentials given by Eq. (90) and $U(\epsilon) = \epsilon^2/2r_q$, respectively, subjected to additive noise. In contrast, noise in the model B is generated by a motion under the potential of $U(\epsilon) = \epsilon^2/2$ subjected to additive and multiplicative noises.

We note from Eqs. (4) and (67) that the stationary distributions of ϵ in the models A_0 and B become the equivalent non-Gaussian distribution if the parameters in the two models satisfy the relation:

$$\phi^2 = \kappa\beta^2 = \frac{\beta^2}{(1+\alpha^2/2\tau)}. \qquad \text{for } q \geq 1 \tag{91}$$

This equivalence, however, does no hold between the models A and B, because the stationary distribution of the model A is not the non-Gaussian but the Gaussian given by

$$p_q(\epsilon) \propto \exp\left[-\left(\frac{\tau}{r_q\phi^2}\right)\epsilon^2\right]. \qquad \text{(model A)} \tag{92}$$

As for the dynamical properties, equations of motion for $\langle\epsilon^2\rangle$ in the moment method are given by Eqs. (17) and (A6):

$$\frac{d\langle\epsilon^2\rangle}{dt} = -\left(\frac{2}{r_q\tau}\right)\langle\epsilon^2\rangle + \left(\frac{\phi}{\tau}\right)^2, \qquad \text{(model A)} \tag{93}$$

$$\frac{d\langle\epsilon^2\rangle}{dt} = -\frac{2}{\tau}\left(1-\frac{\alpha^2}{\tau}\right)\langle\epsilon^2\rangle + \left(\frac{\beta}{\tau}\right)^2, \qquad \text{(model B)} \tag{94}$$

Equations of motion for μ and γ are given by Eqs. (33), (34), (A16) and (A17):

$$\frac{d\mu}{dt} = f_0 + f_2\gamma + I(t), \qquad \text{(models A and B)} \tag{95}$$

$$\frac{d\gamma}{dt} = 2f_1\gamma + \frac{r_q^2\phi^2}{(1-r_q\tau f_1)}, \qquad \text{(model A)} \tag{96}$$

$$\frac{d\gamma}{dt} = 2f_1\gamma + \frac{u_q^2\,\beta^2}{(1-\tau f_1)}, \qquad \text{(model B)} \tag{97}$$

where $u_q = \sqrt{(3-q)/(5-3q)}$ [Eq. (A18)]. In the model A, we have adopted the approximation: $K(\epsilon) \simeq -\epsilon/r_q$ [Eq. (9)], without which reasonable results are not obtainable in the moment approach (see the discussion below). Equations (93)-(97) show that equations of motion for the models A and B have the same structure. In the case of weak noise and small τ, for which the second-moment approach is expected to be valid, the dynamical properties of the models A and B (as well as the model A_0) are qualitatively the same, although there are some quantitative difference among them: *i.e.* the stationary value of γ of the model A is different from that of the model B.

Our discussion presented in this paper is based on the second-order moment method. Effects of higher-order moment neglected in our method are examined in the following. The equation of motion for the kth moment with even k of the model A_0 is formally given by

$$\frac{d\langle\epsilon^k\rangle}{dt} = \left(\frac{k}{\tau}\right)\langle\epsilon^{(k-1)}K(\epsilon)\rangle + \frac{k(k-1)}{2}\left(\frac{\phi}{\tau}\right)^2\langle\epsilon^{(k-2)}\rangle, \qquad \text{(model } \mathrm{A}_0\text{)} \tag{98}$$

though an evaluation of the first term of Eq. (98) is very difficult. In order to get a meaningful result within the moment method, we have assumed $K(\epsilon) \simeq -\epsilon/r_q$ (the model A) to get

$$\frac{d\langle\epsilon^k\rangle}{dt} = -\left(\frac{k}{r_q\tau}\right)\langle\epsilon^k\rangle + \frac{k(k-1)}{2}\left(\frac{\phi}{\tau}\right)^2\langle\epsilon^{(k-2)}\rangle, \qquad \text{(model A)} \tag{99}$$

from which we may recurrently calculate the stationary kth moment as

$$\langle \epsilon^k \rangle = \frac{(k-1)r_q}{2}\left(\frac{\phi^2}{\tau}\right)\langle \epsilon^{(k-2)} \rangle, \tag{100}$$

$$= \frac{(k-1)!!\, r_q^{k/2}}{2^{k/2}}\left(\frac{\phi^2}{\tau}\right)^{k/2}. \qquad \text{(model A)} \tag{101}$$

The stationary distribution in the model A_0 given by Eq. (4) leads to the second- and fourth-order moments:

$$\langle \epsilon^2 \rangle = \frac{\phi^2}{(5-3q)\tau}, \tag{102}$$

$$\langle \epsilon^4 \rangle = \frac{3\phi^4}{(5-3q)(7-5q)\tau^2}. \qquad \text{(model } A_0\text{)} \tag{103}$$

In contrast, the stationary distribution in the model A given by Eq. (92) yields

$$\langle \epsilon^2 \rangle = \frac{(2-q)\phi^2}{(5-3q)\tau}, \tag{104}$$

$$\langle \epsilon^4 \rangle = 3\langle \epsilon^2 \rangle^2 = \frac{3(2-q)^2\phi^4}{(5-3q)^2\tau^2}. \qquad \text{(model A)} \tag{105}$$

The expression of Eq. (104) is different from that of Eq. (102) by a factor of $(2-q)$. The ratio of Eq. (105) to Eq. (103) becomes $(2-q)^2(7-5q)$, which is less than unity for $1 < q < 5/3$. These show that the distribution given by Eq. (92) in the model A underestimates the effective width of the distribution of ϵ compared to that in the model A_0. In order to include the higher-order moment in an appropriate way, we have to go beyond the approximation with $K(\epsilon) \simeq -\epsilon/r_q$ [Eq. (9)].

With the model B, we may obtain the equation of motion for the kth moment with even k, as given by

$$\frac{d\langle \epsilon^k \rangle}{dt} = -\left[\frac{k}{\tau} - \frac{k^2}{2}\left(\frac{\alpha}{\tau}\right)^2\right]\langle \epsilon^k \rangle + \frac{k(k-1)}{2}\left(\frac{\beta}{\tau}\right)^2\langle \epsilon^{(k-2)} \rangle. \qquad \text{(model B)} \tag{106}$$

The stationary value of the kth moment is given by

$$\langle \epsilon^k \rangle = \frac{(k-1)\beta^2}{2(\tau - k\alpha^2/2)}\langle \epsilon^{(k-2)} \rangle, \tag{107}$$

$$= \frac{(k-1)!!\, \beta^k}{2^{k/2}\,\Pi_{\ell=1}^{k/2}(\tau - \ell\alpha^2)}. \tag{108}$$

For example, second- and fourth-moments are given by

$$\langle \epsilon^2 \rangle = \frac{\beta^2}{2(\tau-\alpha^2)}, \tag{109}$$

$$\langle \epsilon^4 \rangle = \frac{3\beta^4}{4(\tau-2\alpha^2)(\tau-\alpha^2)}, \qquad \text{(model B)} \tag{110}$$

which agree with the result obtained from the stationary distribution given by Eq. (66) or (67). We get the positive definite $\langle \epsilon^k \rangle$ for $\alpha^2 < 2\tau/k$. This suggests that for $2\tau/k < \alpha^2 < \tau$ with $k \geq 4$, the kth moment diverges even if the second moment remains finite. This might throw some doubt on the validity of the second-moment approach. Equation (106) expresses that the motion of $\langle \epsilon^k \rangle$ depends on those of its lower moments ($\leq k-2$), but it is independent of its higher moments ($\geq k+2$). For example, even if $\langle \epsilon^4 \rangle$ diverges, it has no effects on the motion of $\langle \epsilon^2 \rangle$ for $\tau/2 < \alpha^2 < \tau$. It is promising to take into account contributions from higher-order moments in the model B, although its validity range becomes narrower because α has to satisfy the condition: $\alpha^2 < 2\tau/k$ for the kth moment to remain finite.

Our discussions presented in the preceding sections are confined to the stationary properties of the Langevin model subjected to non-Gaussian noise. It is possible to discuss its dynamics, by solving equations of motion for μ and γ. Numerical calculations for the model B are plotted in Figure 6(a) and 6(b), which show the time dependences of μ and γ, respectively. We apply an external pulse input given by $I(t) = A\,\Theta(t-100)\Theta(200-t)$ with $A = 1.0$, which is plotted at the bottom of Fig. 6(a), $\Theta(x)$ denoting the Heaviside function. Figure 6(a) shows that $\mu(t)$ of the moment method is in good agreement with the result of DS. Figure 6(b) shows that $\gamma(t)$ is independent of an input pulse [Eq. (A17)]. With increasing α, a steady value of γ is increased. The result of the moment method for $\alpha = 0.0$ is in good agreement with that of DS although for $\alpha = 0.5$, the former is underestimated compared to the latter.

The overall behavior of the stationary distribution is fairly well reproduced by all the approximations mentioned in Sec. 2.2. Tails of $P(x)$ are, however, not satisfactorily described, in particular, in calculations of the model A. This is partly due to the fact that the approximate Eq. (10) yields the Gaussian stationary distribution given by Eq. (92), though Eqs. (2) and (3) are originally introduced to generate non-Gaussian noise. This point is improved in the model B, in which the stationary distribution given by Eq. (64) is non-Gaussian as expressed by Eqs. (66) and (67). Indeed, tails of $P(x)$ of Fig. 5 for the model B are slightly well reproduced than those of Fig. 3 for the model A.

5 Conclusion

To summarize, we have studied effects of non-Gaussian noise on the Langevin model, by using the second-order moment approach. The obtained result is summarized as follows.

(1) With increasing τ, the width of the stationary distribution $P(x)$ is decreased.
(2) For $q > 1$ ($q < 1$), the width of $P(x)$ is increased (decreased) compared to that for $q = 1$.
(3) The prefactor of F_{eff} for the model A in the moment method agrees with that in FI-1, but disagrees with that in the UCNA and FI-2 (Table 1).

The items (1) and (2) are realized in both the models A and B. This may be explained by the q- and τ-dependent α_{eff} [Eq. (38)] or β_{eff} [Eq. (78)]. As for the item (3), it is necessary to point out that although the UCNA [6, 7] exactly interpolates between the two limits of $\tau = 0$ and $\tau = \infty$, it is not exact for $O(\tau)$ [19]. The functional integral method is a formally exact transformation if the functional integral is correctly performed. In the actual applications, however, it is inevitable to adopt some kinds of approximation, with which the final result depends on the adopted approximation. The difference between the results of FI-1 [9] and FI-2 [4] arise from the difference between the adopted approximations in performing the functional integral. These yield the difference in the results listed in the Table 1.

As for the models A and B, we get
(i) although the stationary distribution of $p(\epsilon)$ in the model A is the Gaussian, the effect of the non-Gaussian distribution of the original model A_0 is fairly well taken into account by a factor of r_q, and
(ii) the newly introduced model B, which yields the stationary non-Gaussian $p(\epsilon)$ equivalent to that of the model A_0, is expected to be a promising model generating non-Gaussian noise.

It is possible to apply the moment approach to a wide class of stochastic systems subjected to non-Gaussian noise, because its calculation is simple and transparent. It would be interesting to investigate effects of non-Gaussian noise on the synchronization in coupled nonlinear systems with the use of the model B, which is left as our future study.

Acknowledgements

This work is partly supported by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

Appendix

We discuss an application of the moment method to the model B given by Eqs. (63) and (64), for which the FPE of the distribution $p(x,\epsilon,t)$ in the Stratonovich representation is given by

$$\begin{aligned}\frac{\partial}{\partial t}p(x,\epsilon,t) &= -\frac{\partial}{\partial x}\{[F(x)+\epsilon+I]p(x,\epsilon,t)\}+\frac{1}{\tau}\frac{\partial}{\partial \epsilon}[\epsilon p(x,\epsilon,t)] \\ &+ \frac{1}{2}\left(\frac{\alpha}{\tau}\right)^2\frac{\partial}{\partial\epsilon}\left[\epsilon\frac{\partial}{\partial\epsilon}\epsilon p(x,\epsilon,t)\right]+\frac{1}{2}\left(\frac{\beta}{\tau}\right)^2\frac{\partial^2}{\partial\epsilon^2}p(x,\epsilon,t). \qquad \text{(A1)}\end{aligned}$$

We define means, variances and covariances by

$$\langle x^m\epsilon^n\rangle = \int dx\int d\epsilon\; x^m\epsilon^n p(x,\epsilon,t). \qquad (m,n\text{: integer}) \qquad \text{(A2)}$$

After simple calculations using Eqs. (A1) and (A2), we get their equations of motion given by [13, 14]

$$\begin{aligned}\frac{d\langle x\rangle}{dt} &= \langle F(x)+\epsilon+I\rangle, &\text{(A3)}\\ \frac{d\langle \epsilon\rangle}{dt} &= -\frac{1}{\tau}\langle\epsilon\rangle+\frac{1}{2}\left(\frac{\alpha}{\tau}\right)^2\langle\epsilon\rangle, &\text{(A4)}\\ \frac{d\langle x^2\rangle}{dt} &= 2\langle x[F(x)+\epsilon+I]\rangle, &\text{(A5)}\\ \frac{d\langle \epsilon^2\rangle}{dt} &= -\frac{2}{\tau}\langle\epsilon^2\rangle+2\left(\frac{\alpha}{\tau}\right)^2\langle\epsilon^2\rangle+\left(\frac{\beta}{\tau}\right)^2, &\text{(A6)}\\ \frac{d\langle x\epsilon\rangle}{dt} &= \langle\epsilon[F(x)+\epsilon+I]\rangle-\frac{1}{\tau}\langle x\epsilon\rangle. &\text{(A7)}\end{aligned}$$

We will consider the variables of μ, ν, γ, ζ and ϕ defined by Eqs. (19)-(23). Their equations of motion become [13, 14]

$$\begin{aligned}\frac{d\mu}{dt} &= f_0+f_2\gamma+\nu+I(t), &\text{(A8)}\\ \frac{d\nu}{dt} &= -\frac{\nu}{\tau}+\frac{1}{2}\left(\frac{\alpha}{\tau}\right)^2\nu, &\text{(A9)}\\ \frac{d\gamma}{dt} &= 2(f_1\gamma+\phi), &\text{(A10)}\\ \frac{d\zeta}{dt} &= -\frac{2}{\tau}\zeta+2\left(\frac{\alpha}{\tau}\right)^2\zeta+\nu^2\left(\frac{\alpha}{\tau}\right)^2+\left(\frac{\beta}{\tau}\right)^2, &\text{(A11)}\\ \frac{d\phi}{dt} &= \left(-\frac{1}{\tau}+f_1\right)\phi+\zeta-\frac{\mu\nu}{2}\left(\frac{\alpha}{\tau}\right)^2, &\text{(A12)}\end{aligned}$$

where $f_\ell=(1/\ell!)\partial^\ell F(\mu)/\partial x^\ell$.

When we assume the stationary values for ν, ζ and ϕ:

$$\nu \simeq \nu_s = 0, \tag{A13}$$
$$\zeta \simeq \zeta_s = \frac{\beta^2}{2(\tau - \alpha^2)}, \tag{A14}$$
$$\phi \simeq \phi_s = \frac{\beta^2}{2(1-\alpha^2/\tau)(1-\tau f_1)}, \tag{A15}$$

equations of motion for μ and γ become

$$\frac{d\mu}{dt} = f_0 + f_2\gamma + I(t), \tag{A16}$$
$$\frac{d\gamma}{dt} = 2f_1\gamma + \frac{u_q^2\,\beta^2}{(1-\tau f_1)}, \tag{A17}$$

with

$$u_q = \sqrt{\frac{1}{(1-\alpha^2/\tau)}} = \sqrt{\frac{3-q}{5-3q}}. \tag{A18}$$

With increasing τ, u_q is decreased because of a decreased q (Fig. 1). Equations (A17) and (A18) lead to the stationary value of γ_s given by

$$\gamma_s = \frac{\beta^2(3-q)}{(-2f_1)(1-\tau f_1)(5-3q)} = \frac{\tau}{(-f_1)(1-\tau f_1)}\langle\epsilon^2\rangle. \tag{A19}$$

It is noted that equations of motion given by Eqs. (A16) and (A17) may be derived from the one-variable DE given by Eq. (76) [13, 14].

F_{eff}	I_{eff}	α_{eff}	method
F	I	$r_q\phi/(\sqrt{1-r_q\tau f_1})$	moment$^{1)}$
F	$-$	$r_q\phi/(\sqrt{1-r_q\tau F_s'})$	FI-1$^{2)}$
$F/(1-r_q\tau F')$	$(I+\tau\dot{I})/(1-r_q\tau F')$	$r_q\phi/(1-r_q\tau F')$	UCNA$^{3)}$
$F/(1-s_q\tau F')$	$-$	$s_q\phi/(1-s_q\tau F')$	FI-2$^{4)}$

Table 1 A comparison among various approaches to the model A [Eqs. (1) and (10)] yielding the effective differential equation given by $\dot{x} = F_{eff} + I_{eff} + \alpha_{eff}\ \eta(t)$, where $r_q = 2(2-q)/(5-3q)$ and $s_q = 1 + (q-1)(\tau/2\phi^2)F^2$; (1) the moment method: (2) functional-integral (FI-1) method of Ref. [9]: (3) UCNA calculation after Ref. [6, 7]: (4) functional-integral (FI-2) method of Ref. [4] (see text).

References

[1] K. Wiesenfeld, D. Pierson, E. Pantazelou, Ch. Dames, and F. Moss, Phys. Rev. Lett. **72**, 2125 (1994).

[2] D. Nozaki, D. J. Mar, P. Grigg, and J. J. Collins, Phys. Rev. Lett. **82**, 2402 (1999).

[3] L. Borland, Phys. Lett. A **245**, 67 (1998).

[4] M. A. Fuentes, R. Toral, and H. S. Wio, Physica A **295**, 114 (2001).

[5] F. J. Castro, M. N. Kuperman, M. Fuentes, and H. S. Wio, Phys. Rev. E **64**, 051105 (2001).

[6] P. Jung and P. Hänggi, Phys. Rev. A **35**, 4464 (1987).

[7] P. Hänggi and P. Jung, Adv. Chem. Phys. **89**, 239 (1995).

[8] M. A. Fuentes, R. Toral, and H. S. Wio, Physica A **303**, 91 (2002).

[9] D. Wu, X. Luo, and S. Zhu, Physica A **373**, 203 (2007).

[10] H. Sakaguchi: J. Phys. Soc. Jpn. **70**, 3247 (2001).

[11] C. Anteneodo and C. Tsallis, J. Math. Phys. **44**, 5194 (2003).

[12] C. Anteneodo and R. Riera, Phys. Rev. E **72**, 026106 (2005).

[13] H. Hasegawa, J. Phys. Soc. Jpn. **75**, 033001 (2006).

[14] H. Hasegawa, Physica A **374**, 585 (2007).

[15] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[16] C. Tsallis, R. S. Mendes, and A. R. Plastino, Physica A **261**, 534 (1998).

[17] E. A. Novikov, Soviet Phys. JETP **20**, 1290 (1965).

[18] After taking the derivative of Eq. (63) for the model B with respect to t and eliminating the variable ϵ with the use of Eq. (64), we get $\ddot{x} = (F' - 1/\tau)\dot{x} + (F + I)/\tau + (\dot{x} - F - I)(\alpha/\tau)\eta + (\beta/\tau)\xi$. The third (multiplicative-noise) term includes functions of x and $\dot{x}$, which prevents an application of the UCNA [6, 7] to the model B.

[19] K. Mallick and P-E. Peyneau, Physica D **221**, 72 (2006).

Figure 1: The q dependence of r_q [Eq. (11): solid curve] and u_q [Eq. (80): dashed curve].

Figure 2: The stationary probability $P(x)$ for $F(x) = -\lambda x$ of the model A_0 [Eqs. (1)-(3)] calculated by direct simulation (DS: dashed curves), and $P(x)$ of the model A [Eqs. (1) and (10)] calculated by the moment method (solid curves), UCNA (chain curves) and FI-2 (dotted cures): results of FI-1 agree with those of the moment method: (a) $(q, \tau) = (0.8, 0.5)$, (b) $(0.8, 1.0)$, (c) $(1.0, 0.5)$, (d) $(1.0, 1.0)$, (e) $(1.5, 0.5)$, and (f) $(1.5, 1.0)$ ($\phi = 0.5$ and $\psi = 0.0$).

Figure 3: The stationary probability $P(x)$ for $F(x) = x - x^3$ of the model A_0 [Eqs. (1)-(3)] calculated by direct simulation (DS: dashed curves), and $P(x)$ of the model A [Eqs. (1) and (10)] calculated by the moment method (solid curves), UCNA (chain curves) and FI-2 (dotted cures): results of FI-1 agree with those of the moment method: (a) $(q, \tau) = (0.8, 0.5)$, (b) $(0.8, 1.0)$, (c) $(1.0, 0.5)$, (d) $(1.0, 1.0)$, (e) $(1.5, 0.5)$, and (f) $(1.5, 1.0)$ ($\phi = 0.5$ and $\psi = 0.0$).

Figure 4: The stationary probability $P(x)$ for $F(x) = -\lambda x$ of the model B given by Eqs. (63) and (64) with (a) $(\alpha, \tau) = (0.0, 0.5)$ ($q = 1.40$), (b) $(0.0, 1.0)$ ($q = 1.22$), (c) $(0.5, 0.5)$ ($q = 1.40$), and (d) $(0.5, 1.0)$ ($q = 1.22$), calculated for $\beta = 0.5$ by DS (dashed curves) and the moment method (solid curves).

Figure 5: The stationary probability $P(x)$ for $F(x) = x - x^3$ of the model B given by Eqs. (63) and (64) with (a) $(\alpha, \tau) = (0.0, 0.5)$ ($q = 1.40$), (b) $(0.0, 1.0)$ ($q = 1.22$), (c) $(0.5, 0.5)$ ($q = 1.40$), and (d) $(0.5, 1.0)$ ($q = 1.22$), calculated for $\beta = 0.5$ by DS (dashed curves) and the moment method (solid curves).

Figure 6: The time dependence of (a) $\mu(t)$ and (b) $\gamma(t)$ of the model B for $\alpha = 0.0$ and $\alpha = 0.5$ with $\beta = 0.5$ and $\tau = 1.0$, calculated by the moment method (solid curves) and DS (dashed curves), an input pulse being plotted at the bottom of (a). Results for $\alpha = 0.5$ in (a) is indistinguishable from those for $\alpha = 0.0$.